Partitioning 3-coloured complete graphs into three monochromatic paths

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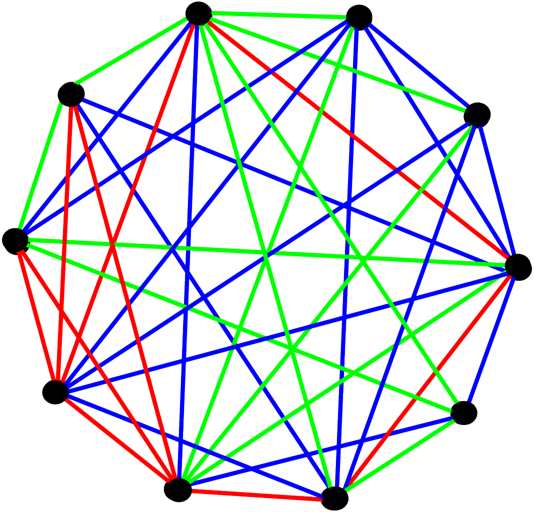

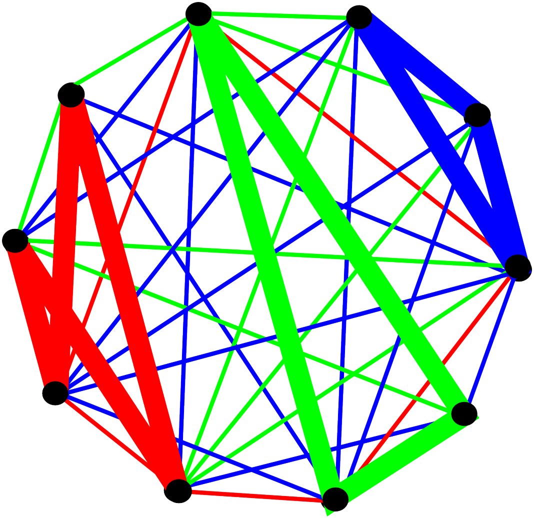

Ramsey Theory

The Ramsey Number $R(G, H)$ is the smallest n for which any 2-edge-colouring of K_n contains either a red G or a blue H.

Theorem (Ramsey, 1930)

$R(K_n, K_n)$ is finite for every n.

The following bounds hold

$$\sqrt{2}^n \leq R(K_n, K_n) \leq 4^n.$$

Theorem (Gerencsér and Gyárfás, 1966)

For $m \leq n$ we have that

$$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

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THEOREM 1. *For $k \geq l$ we have*

(1) $$g(k,l) = k + \left[\frac{l+1}{2}\right].$$

Considering the other special case of this type of problems, let $f_r(n)$ denote the greatest integer with the property, that colouring the edges of a complete n-tuple g with r colours arbitrarily, there exists always a one-coloured connected subgraph with at least $f_r(n)$ vertices.

It is easy to see the following remark of P. ERDŐS: if a graph is not connected then its complement is connected, i.e. $f_2(n) = n$. We shall prove

THEOREM 2.

(2) $$f_3(n) = \left[\frac{n+1}{2}\right].$$

Now we turn to the proof of Theorem 1. First we prove $g(k,l) \leq k + \left[\frac{l+1}{2}\right]$ by induction on k. For $k=1$ the Theorem evidently holds and let us suppose that for all k-s less than this the statement is true. Let us consider a graph G with $k+\left[\frac{l+1}{2}\right]$ vertices. If $l<k$, then for any subgraph of G with $k-1+\left[\frac{l+1}{2}\right]$ points holds that either itself contains a path of length $k-1$, or its complement a path of length l. For $l=k$ we consider a subgraph with $k-1+\left[\frac{l}{2}\right]$ points. This or its complement contains a path of length $k-1$. Thus in every case can be supposed, that the length of the longest path of G is $k-1$. Let $U_1, U_2, \dots, U_k$ be the consecutive vertices of such a path and $U = \{U_1, \dots, U_k\}$. We denote the remaining vertices by $V_1, \dots, V_{\left[\frac{l+1}{2}\right]}$ and the set of them by $V = \left\{V_1, \dots, V_{\left[\frac{l+1}{2}\right]}\right\}$.

It clearly holds that

(i) for all $V_i \in V$ either $V_iU_j \in \bar G$ or $V_iU_{j+1} \in \bar G$

(ii) for all $V_i \in V$ $V_iU_1 \in \bar G$ and $V_iU_k \in \bar G$

(iii) for $V_{i1}, V_{i2}, V_{i3} \in V$ and $U_j, U_{j+1} \in U$

at least one of the latest points is connected in $\bar G$ with at least two of V_{i1}, V_{i2}, V_{i3}.

Consider a maximal path of $\bar G$ not containing U_1, U_k with the property that any edge of it connects a point of U with a point of V, and its endpoints are in V; let us denote the endpoints by A and B, and the path by S. If S contains all points of V, then by adding the edges U_1A, BU_k we have a path of length $2\left[\frac{l+1}{2}\right] \geq l$ in $\bar G$. So we may suppose that the set of points V not contained by S is not empty. Let this set be called W. Consider a maximal path q of $\bar G$ not containing U_1, U_k and having no common points with S, such that any edge of it connects a point of U with a point of W and the endpoints of it,

called by C and D, are in W. We show that all points of V are contained either in S or in q. Suppose that $X \in V$ but $X \notin S$, $X \notin q$. It is clear, that the number of vertices of S and q in U is at most $\left[\frac{l+1}{2}\right]-3 < \left[\frac{k-3}{2}\right] = \left[\frac{k-2-1}{2}\right]$ since $l \leq k$. So there exist two points $U_i, U_{i+1} \in \{U_2, \dots, U_{k-1}\}$ which do not belong either to S or to q. Applying (iii) for $A, C, X \in V$ and $U_i, U_{i+1} \in U$ we have a contradiction to the maximal properties of S and q.

So the sum of the length of S and q is $2\left[\frac{l+1}{2}\right]-4$. We add them the edges U_1A, BU_k, U_kC, DU_1 and so we have a circuit of length $2\left[\frac{l+1}{2}\right]$ in $\bar G$. For odd l this contains a desired path with length l. For even l an easy reasoning shows that there are $U_i, U_{i+1} \in U$ which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length l in $\bar G$. That completes the proof.

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b) For even l we can leave one of the edges of H_1. These graphs possess obviously the desired property.[1]

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Partitioning coloured graphs

Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

Conjecture (Lehel, 1979)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic cycles with different colours.

Single edge or single vertex count as cycles.

Conjecture (Erdős, Gyárfás, and Pyber, 1991)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic cycles.

Results for arbitrarily many colours

Theorem (Erdős, Gyárfás, Pyber, 1991)

There exists a function $f(r)$ such that any r-edge-coloured K_n can be covered by $f(r)$ disjoint monochromatic cycles.

- Erdős, Gyárfás and Pyber proved this theorem with $f(r) = O(r^2 \log r)$.
- Gyárfás, Ruszinkó, Sárközy and Szemerédi improved the bound to $f(r) = O(r \log r)$.
- Major open problem to show $f(r) \leq Cr$ for some C.

Two colours

Suppose that the edges of K_n are coloured with 2 colours...

- There exists a covering with 2 disjoint monochromatic paths. [Gerencsér, Gyárfás, 1967]
- There exists a covering of K_n by 2 monochromatic cycles, intersecting in at most one vertex. [Gyárfás, 1983]
- If n is very large, there exists a covering by 2 disjoint monochromatic cycles. [Łuczak, Rödl, Szemerédi, 1998]
- If n is large, there exists a covering of K_n by 2 disjoint monochromatic cycles. [Allen, 2008]
- There exists a covering of K_n by 2 disjoint monochromatic cycles. [Bessy, Thomassé, 2010]

Three colours

Theorem (Gyárfás, Ruszinkó, Sárközy, Szemerédi, 2011)

Every 3-edge-coloured K_n contains 3 disjoint monochromatic cycles covering $n - o(n)$ vertices.

Theorem (P., 2013+)

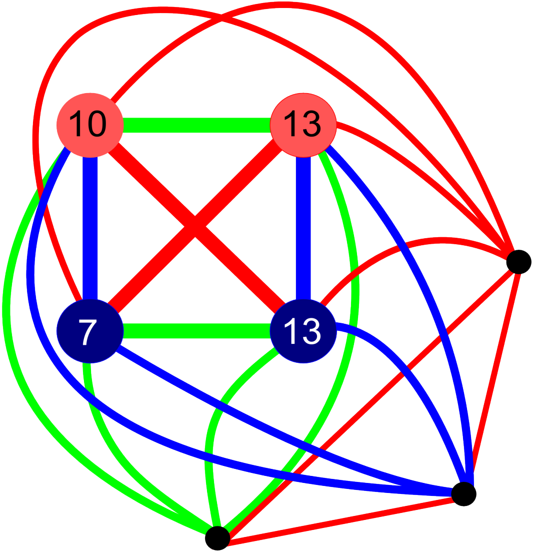
For every $r \geq 3$, and $n \geq N_r$ there exists an r-edge-coloured of K_n which cannot be covered by r disjoint monochromatic cycles.

Theorem (P., 2013+)

There is a constant c such that every 3-edge-coloured K_n contains 3 disjoint monochromatic cycles covering $n - c$ vertices.

Counterexamples

A 3-edge-coloured K_{47} which cannot be covered by 3 disjoint monochromatic cycles.



Covering a 3-coloured complete graph by 3 cycles

Theorem (P., 2013+)

There is a constant c such that every 3-edge-coloured K_n contains 3 disjoint monochromatic cycles covering $n - c$ vertices.

Proof is based on two lemmas.

Lemma

Let K_n be a 2-edge-coloured complete graph such that the red colour class is k-connected. Then K_n can be covered by a red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{k}{k+1}|H| - 4.$$

Lemma

There exist constants $\epsilon > 0$ and c such that every 2-edge-coloured graph G with minimum degree $1 - \epsilon)|G|$ contains two disjoint

Partitioning a graph into a cycle and a sparse graph

Lemma

Let K_n be a 2-edge-coloured complete graph such that the red colour class is k-connected. Then K_n can be covered by a red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{k}{k+1}|H| - 4.$$

- The constant "$\frac{k}{k+1}$" is best possible.
- The constant "-4" is not.

Lemma

Every 2-edge-coloured K_n can be covered by red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{1}{2}|H| - \frac{1}{2}.$$

Somewhat annoyingly, the constant "$-1/2$" is best possible.

Open problems

Conjecture (Gyárfás)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic **paths**.

True for $r = 2$ and 3.

Conjecture

For each r there exists a constant c_r such that every r-edge-coloured complete graph K_n contains r disjoint monochromatic cycles on $n - c_r$ vertices.

Conjecture

Every r-edge-coloured complete graph can be covered by r (not necessarily disjoint) monochromatic cycles.

Open problems

Conjecture (Gyárfás)

Let G be a 2-edge-coloured graph with minimum degree δ.

(i) $\delta > \frac{3}{4} \implies$ *G can be covered by 2 disjoint monochromatic cycles.*

(ii) $\delta > \frac{2}{3} \implies$ *G can be covered by 3 disjoint monochromatic cycles.*

(iii) $\delta > \frac{1}{2} \implies$ *G can be covered by 4 disjoint monochromatic cycles.*

Part (i) was conjectured separately by Balogh, Barát, Gerbner, Gyárfás & Sárközy.

Open problems

Lemma

Every 2-edge-coloured K_n can be covered by red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{1}{2}|H| - \frac{1}{2}.$$

Problem

Prove natural statements of the form "Every 2-edge-coloured complete graph can be covered by a red graph G and a disjoint blue graph H with G and H having particular structures".

Known results of this type:

- G and H paths (Gerencsér and Gyárfás).
- G and H cycles (Łuczak, Rödl, and Szemerédi; Allen; Bessy and Thomassé).
- G a matching, H a complete graph (folklore).