# Partitioning 3-coloured complete graphs into three monochromatic paths

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Covering Coloured Graphs by Cycles

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The Ramsey Number R(G, H) is the smallest *n* for which any 2-edge-colouring of  $K_n$  contains either a red *G* or a blue *H*.

Theorem (Ramsey, 1930)

 $R(K_n, K_n)$  is finite for every n.

The following bounds hold

$$\sqrt{2}^n \leq R(K_n, K_n) \leq 4^n.$$

Theorem (Gerencsér and Gyárfás, 1966) For  $m \le n$  we have that

$$R(P_n,P_m)=n+\left\lfloor\frac{m}{2}\right\rfloor-1.$$

< (<sup>1</sup><sup>1</sup>) ▶

#### L. GERENCSÉR AND A. GYÁRFÁS

THEOREM 1. For  $k \ge l$  we have

 $g(k, l) = k + \left[\frac{l+1}{2}\right].$ Considering the other special case of this type of problems, let  $f_n(n)$  denote the greatest integer with the property, that colouring the edges of a complete n-tuple g with r colours arbitrarily, there exists always a one-coloured connected subgraph with at least  $f_i(n)$  vertices.

It is easy to see the following remark of P. ERDOS: if a graph is not connected then its complement is connected, i.e.  $f_0(n) = n$ . We shall prove

THEOREM 2.

 $f_3(n) = \left[\frac{n+1}{2}\right]$ 

Now we turn to the proof of Theorem 1. First we prove  $g(k, l) \equiv k + l$ 

by induction on k. For k = 1 the Theorem evidently holds and let us suppose that for all k-s less than this the statement is true. Let us consider a graph G with  $k + \left\lfloor \frac{l+1}{r} \right\rfloor$  vertices. If l < k, then for any subgraph of G with k-1+points holds that either itself contains a path of length k-1, or its complement a path of length l. For l=k we consider a subgraph with k-1+points.

This or its complement contains a path of length k-1. Thus in every case can be supposed, that the length of the longest path of G is k-1. Let  $U_1, U_2, \ldots, U_k$ be the consecutive vertices of such a path and  $U = \{U_1, \ldots, U_k\}$ . We denote the remaining vertices by  $V_1, \ldots, V_{\lceil l+1 \rceil}$  and the set of them by V =

$$= \left\{ V_1, \ldots, V_{\left[\frac{l+1}{2}\right]} \right\}.$$

It clearly holds that

(i) for all  $V_i \in V$  either  $V_i U_i \in \overline{G}$  or  $V_i U_{i+1} \in \overline{G}$ (ii) for all  $V_i \in V$   $V_i U_i \in \overline{G}$  and  $V_i U_k \in \overline{G}$ (iii) for  $V_{i3}$ ,  $V_{i2}$ ,  $V_{i3} \in V$  and  $U_{i3}$ ,  $U_{i+1} \in U$ 

at least one of the latest points is connected in G with at least two of Vn, Vm, Vm,

Consider a maximal path of  $\overline{G}$  not containing  $U_1, U_k$  with the property that any edge of it connects a point of U with a point of V, and its endpointsare in V: let us denote the endpoints by A and B, and the path by S. If S con tains all points of V, then by adding the edges  $U_1A$ ,  $BU_k$  we have a path of length  $2\left[\frac{l+1}{l}\right] \ge l$  in  $\overline{G}$ . So we may suppose that the set of points V not contained by S is not empty. Let this set be called W. Consider a maximal path g of  $\overline{G}$  not containing  $U_1, U_2$  and having no common points with S, such that any edge of it connects a point of U with a point of W and the endpoints of it, ON RAMSEY-TYPE PROBLEMS

called by C and D, are in W. We show that all points of V are contained either in S or in q. Suppose that  $X \in V$  but  $X \notin S$ ,  $X \notin q$ . It is clear, that the number of vertices of S and q in U is at most  $\left|\frac{l+1}{2}\right| - 3 < \left|\frac{k-3}{2}\right| = \left|\frac{k-2-1}{2}\right|$  $l \leq k$ . So there exist two points  $U_i, U_{i+1} \in \{U_2, \dots, U_{k-1}\}$  which do not belong either to S or to q. Applying (iii) for A, C, X \in V and  $U_i, U_{i+1} \in U$  we have a contradiction to the maximal properties of S and q.

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no path of length k, and for them at the same time  $\tilde{G}$  have no path of length l.

a) Let G consist of the disjoint graphs  $H_1$ ,  $H_2$  with k and  $\left\lfloor \frac{l+1}{2} \right\rfloor - 1$  points

respectively, where the graph H, is complete.

b) For even l we can leave one of the edges of  $H_1$ . These graphs possess obviously the desired property.1

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph G into three classes, i.e. let the edges of G be coloured with red, yellow and blue colours. So we get the graphs Gr, Gy and Gh formed by the red. vellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of G., Let us take a maximal red-connected subgraph R. It may be supposed that R is not empty and  $\pi(R) <$  $\prec \pi(G) = n$ . Let B be a point of G such that  $B \notin R$ . Since R is a maximal connected subgraph of G.,  $BR_i$  is not red for  $R_i \in R$ . So one may suppose that

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#### L. GERENCSÉR AND A. GYÁRFÁS

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# Partitioning coloured graphs

## Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

## Conjecture (Lehel, 1979)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic cycles with different colours.

Single edge or single vertex count as cycles.

## Conjecture (Erdős, Gyárfás, and Pyber, 1991)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic cycles.

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# Results for arbitrarily many colours

### Theorem (Erdős, Gyárfás, Pyber, 1991)

There exists a function f(r) such that any r-edge-coloured  $K_n$  can be covered by f(r) disjoint monochromatic cycles.

- Erdős, Gyárfás and Pyber proved this theorem with  $f(r) = O(r^2 \log r)$ .
- Gyárfás, Ruszinkó, Sárközy and Szemerédi improved the bound to f(r) = O(r log r).
- Major open problem to show  $f(r) \leq Cr$  for some C.

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# Two colours

Suppose that the edges of  $K_n$  are coloured with 2 colours...

- There exists a covering with 2 disjoint monochromatic paths. [Gerencsér, Gyárfás, 1967]
- There exists a covering of K<sub>n</sub> by 2 monochromatic cycles, intersecting in at most one vertex. [Gyárfás, 1983]
- If n is very large, there exists a covering by 2 disjoint monochromatic cycles. [Łuczak, Rödl, Szemerédi, 1998]
- If n is large, there exists a covering of K<sub>n</sub> by 2 disjoint monochromatic cycles. [Allen, 2008]
- There exists a covering of K<sub>n</sub> by 2 disjoint monochromatic cycles.
   [Bessy, Thomassé, 2010]

# Three colours

Theorem (Gyárfás, Ruszinkó, Sárközy, Szemerédi, 2011) Every 3-edge-coloured  $K_n$  contains 3 disjoint monochromatic cycles covering n - o(n) vertices.

#### Theorem (P., 2013+)

For every  $r \ge 3$ , and  $n \ge N_r$  there exists an r-edge-coloured of  $K_n$  which cannot be covered by r disjoint monochromatic cycles.

# Theorem (P., 2013+)

There is a constant c such that every 3-edge-coloured  $K_n$  contains 3 disjoint monochromatic cycles covering n - c vertices.

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# Counterexamples

A 3-edge-coloured  $K_{47}$  which cannot be covered by 3 disjoint



monochromatic cycles.

# Covering a 3-coloured complete graph by 3 cycles Theorem (P., 2013+)

There is a constant c such that every 3-edge-coloured  $K_n$  contains 3 disjoint monochromatic cycles covering n - c vertices.

Proof is based on two lemmas.

#### Lemma

Let  $K_n$  be a 2-edge-coloured complete graph such that the red colour class is k-connected. Then  $K_n$  can be covered by a red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{k}{k+1}|H| - 4.$$

#### Lemma

There exist constants  $\epsilon > 0$  and c such that every 2-edge-coloured graph G with minimum degree  $1 - \epsilon$  ||G| contains two disjoint Alexey Pokrovskiy (Freie) Covering Coloured Graphs by Cycles October 24, 2013 14 / 18

# Partitioning a graph into a cycle and a sparse graph

#### Lemma

Let  $K_n$  be a 2-edge-coloured complete graph such that the red colour class is k-connected. Then  $K_n$  can be covered by a red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{k}{k+1}|H| - 4.$$

- The constant " $\frac{k}{k+1}$ " is best possible.
- The constant "-4" is not.

#### Lemma

Every 2-edge-coloured  $K_n$  can be covered by red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{1}{2}|H| - \frac{1}{2}.$$

# Open problems

# Conjecture (Gyárfás)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic **paths**.

True for r = 2 and 3.

#### Conjecture

For each r there exists a constant  $c_r$  such that every r-edge-coloured complete graph  $K_n$  contains r disjoint monochromatic cycles on  $n - c_r$  vertices.

#### Conjecture

Every r-edge-coloured complete graph can be covered by r (not necessarily disjoint) monochromatic cycles.

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# Open problems

## Conjecture (Gyárfás)

Let G be a 2-edge-coloured graph with minimum degree  $\delta$ .

- (i)  $\delta > \frac{3}{4} \implies G$  can be covered by 2 disjoint monochromatic cycles.
- (ii)  $\delta > \frac{2}{3} \implies G$  can be covered by 3 disjoint monochromatic cycles.
- (iii)  $\delta > \frac{1}{2} \implies G$  can be covered by 4 disjoint monochromatic cycles.

Part (i) was conjectured separately by Balogh, Barát, Gerbner, Gyárfás & Sárközy.

# Open problems

#### Lemma

Every 2-edge-coloured  $K_n$  can be covered by red cycle and a blue graph H satisfying

$$\delta(H) \geq \frac{1}{2}|H| - \frac{1}{2}.$$

#### Problem

Prove natural statements of the form "Every 2-edge-coloured complete graph can be covered by a red graph G and a disjoint blue graph H with G and H having particular structures".

Known results of this type:

- G and H paths (Gerencsér and Gyárfás).
- *G* and *H* cycles (Łuczak, Rödl, and Szemerédi; Allen; Bessy and Thomassé).
- G a matching, H a complete graph (folklore).