Connectedness in tournaments

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Connectedness

Definition

A directed graph is (strongly) connected if for any pair of vertices x and y, there is a directed path from x to y and from y to x.

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A directed graph is (strongly) k-connected if it remains strongly connected after the removal of any set of $k-1$ vertices.

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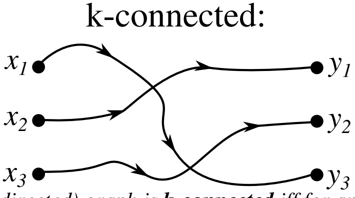
A directed graph is (strongly) k-connected if it remains strongly connected after the removal of any set of $k-1$ vertices.

Theorem (Menger)

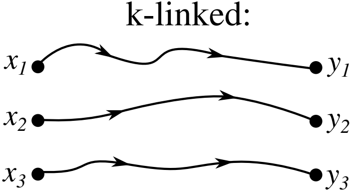
For $n \geq 2k$, a directed graph is k-connected if, and only if, for any two disjoint sets of k vertices S and T, there are k vertex-disjoint paths going from S to T

Linkedness



A (directed) graph is ***k-connected*** *iff for any two disjoint sets of vertices* $\{x_1,...,x_k\}$ *and* $\{y_1,...,y_k\}$ *there are disjoint paths* $P_1,...,P_k$ *such that* P_i *goes from* x_i *to* $y_{p(i)}$ *for some permutation p.*

[Menger's Theorem]



A (directed) graph is ***k-linked*** *if for any two disjoint sets of vertices* $(x_1,...,x_k)$ *and* $(y_1,...,y_k)$ *there are disjoint paths* $P_1,...,P_k$ *such that* P_i *goes from* x_i *to* y_i*.*

[Definition]

Linkedness

Theorem (Lader and Mani; Jung)

There is a function $f(k)$ such that every $f(k)$-connected (undirected) graph is k-linked.

$f(k)$ has been subsequently improved by Mader, Komlós and Szemerédi, and Robertson and Seymour.

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Theorem (Bollobás and Thomason)

Every $22k$-connected (undirected) graph is k-linked.

The constant "22" has been reduced to "10" by Thomas an Wollan.

Linkedness

Theorem (Thomassen)

For every k, there are k-connected directed graphs which are not 2-linked.

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Theorem (Thomassen)

There is a function $f(k)$ such that every $f(k)$-connected tournament is k-linked.

$f(2) = 5$ (Bang-Jensen)

Theorem (Kühn, Lapinskas, Osthus, and Patel)

Every $10^4 k \log k$-connected tournament is k-linked.

The proof uses optimal sorting networks of Ajtai, Komlós and Szemerédi.

Linkedness

Conjecture (Kühn, Lapinskas, Osthus, and Patel)

There is a constant C such that every Ck-connected tournament is k-linked.

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The proof uses "linkage structures" introduced by Kühn, Lapinskas, Osthus, and Patel.

Linkage structures

- Informally a linkage structure L is a small set of vertices in a tournament such that for a pair of vertices x, y in T, there is a path P from x to y, mostly contained in L.

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- We want results of the form "If a tournament is highly connected then it has many disjoint linkage structures".
- The following is the simplest example of such a theorem to state:

Theorem (Kühn, Osthus, and Townsend)

All strongly $10^{16}k^3\log(k^2)$-connected tournaments contain k vertex-disjoint sets $L_1, \ldots, L_k$ with the following property:
For any pair of vertices x and y outside $L_1, \ldots, L_k$ and every i, there is an x to y path contained in $L_i + x + y$

Building linkage structures

Lemma

Every k-connected tournament on $\geq 2k$ vertices contains disjoint sets of vertices $L_1, \ldots, L_k$ with the following property:

For every pair of vertices x and y and every i, there is an x to y path P_i with at most 6 vertices outside L_i

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The proof need the following simple fact.

Fact

Every tournament T with minimum outdegree $\geq k$ contains k vertices $v_1, \ldots, v_k$ (called ***sinks****) such that every vertex in T has a path of length at most 3 to v_i for all i.*

- The outneighbourhood of any vertex of maximum in-degree will satisfy the above fact.
- Similarly one can find **sources** with short paths **to** any vertex.

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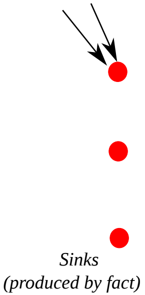

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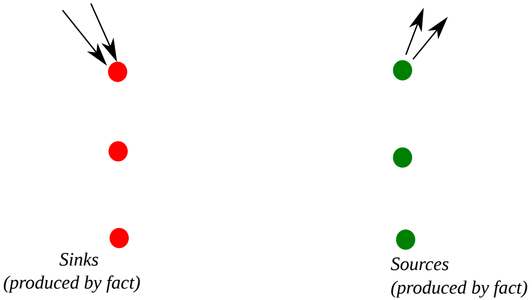

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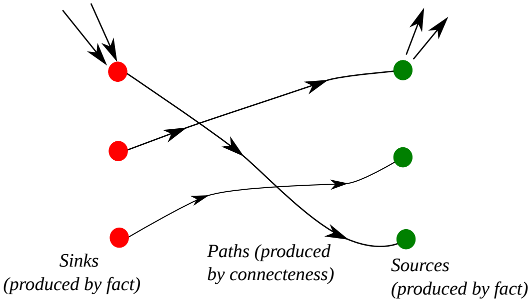

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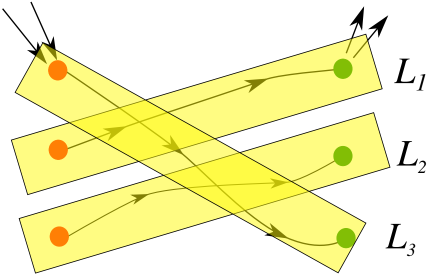

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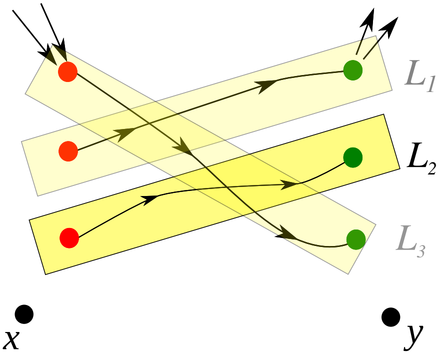

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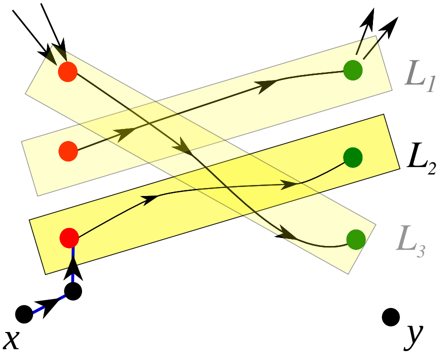

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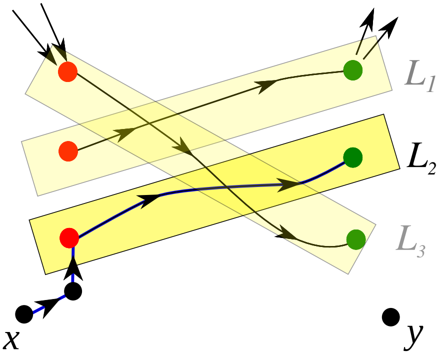

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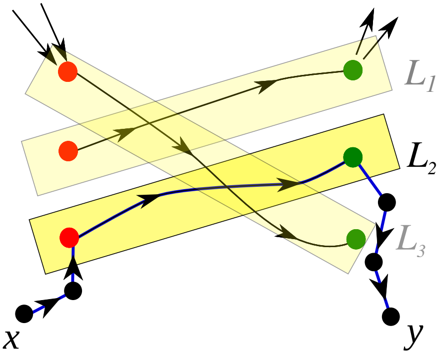

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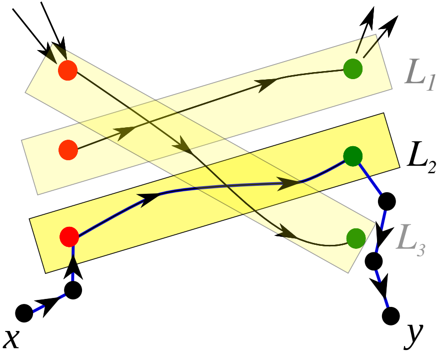

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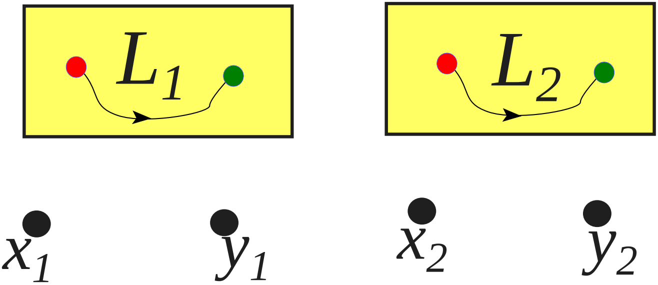

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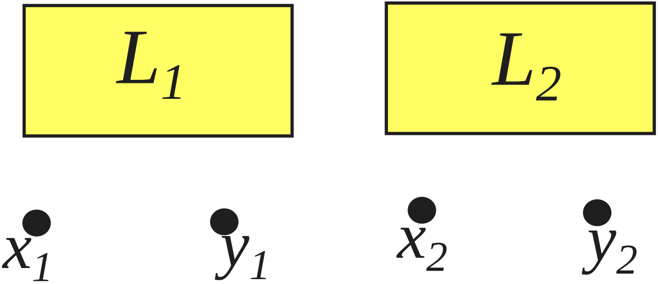

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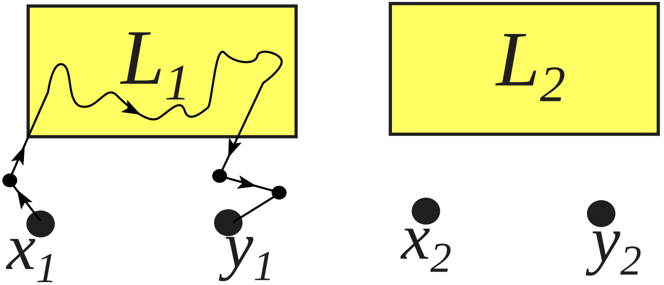

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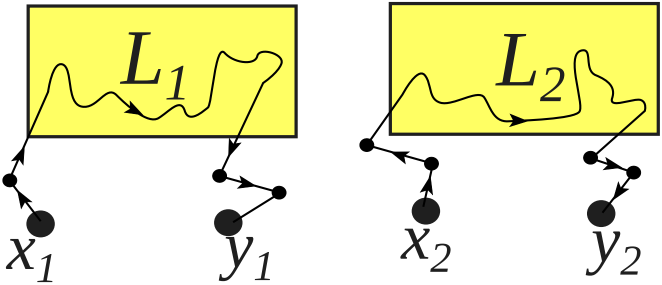

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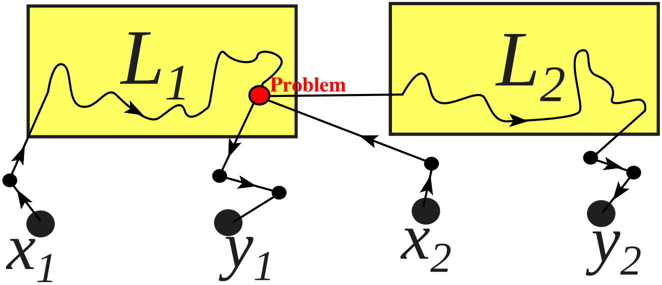
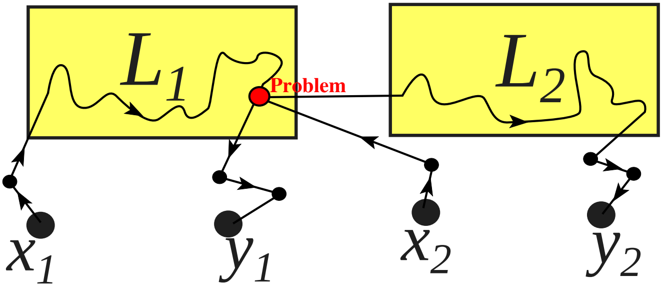


Linkedness

Theorem

Let T be a k-connected tournament with $|T| \geq 2k$. For any two disjoint sets of vertices $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$, there are vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i goes from x_i to y_i, and $|P_i \cap P_j| \leq 12$ for $i \neq j$.



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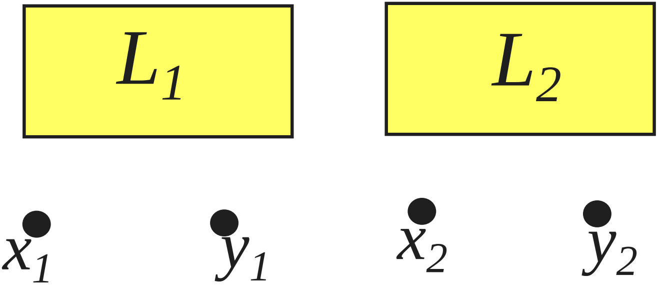

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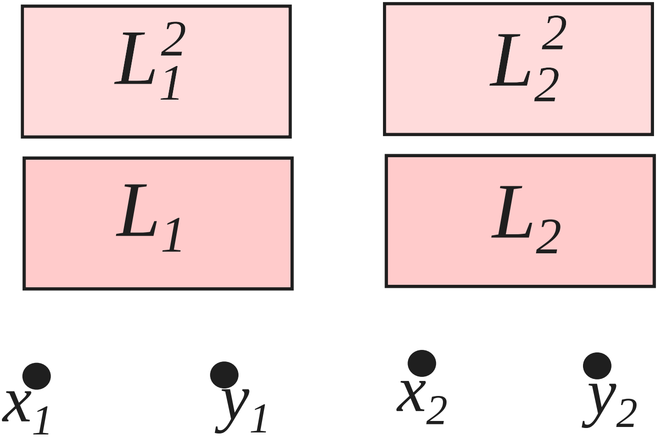

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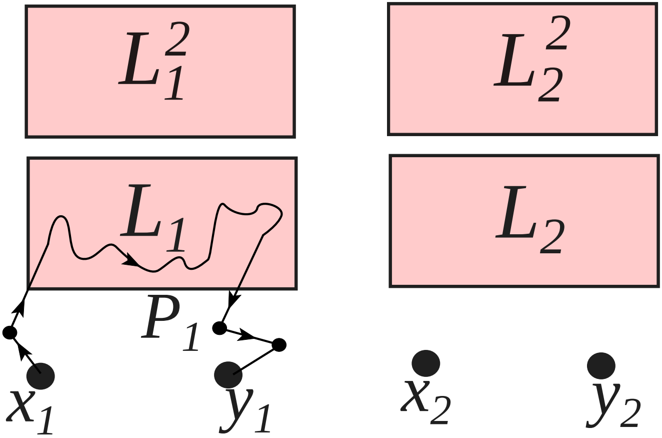

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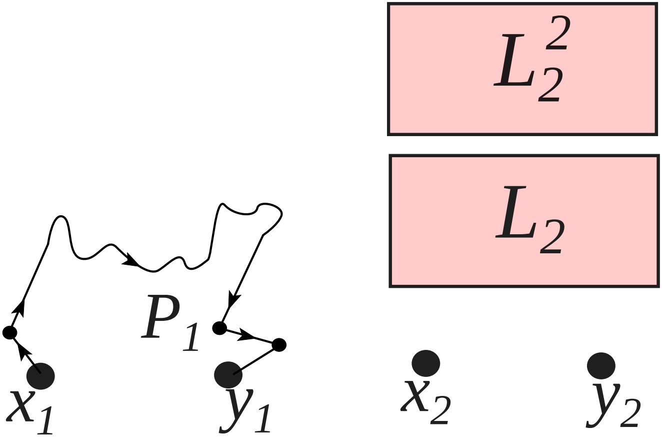

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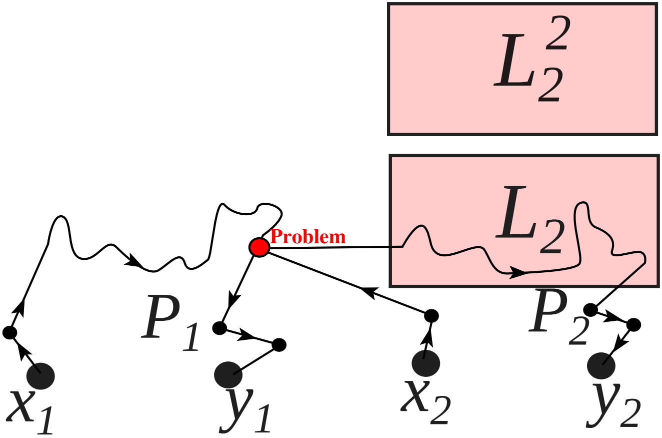

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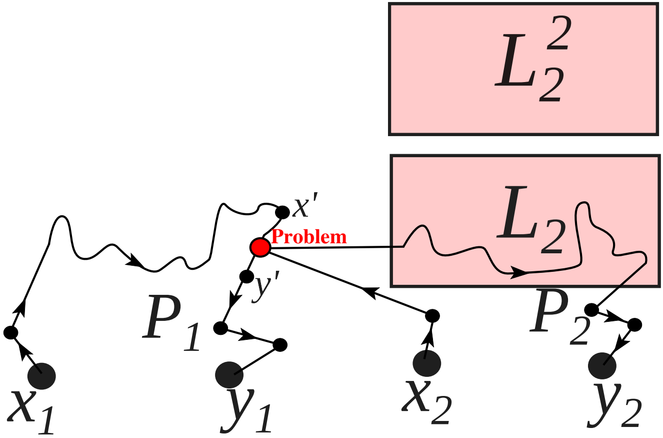

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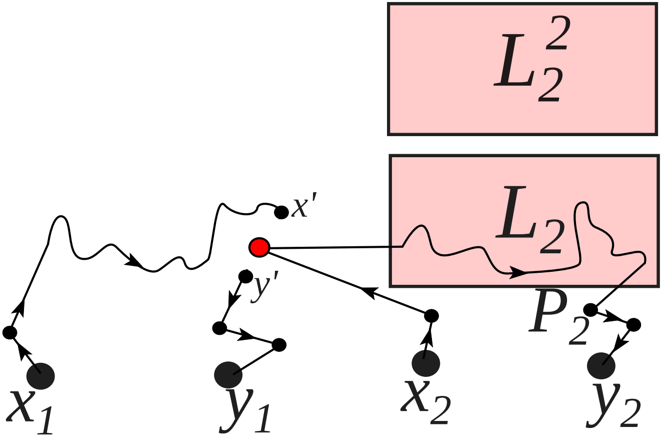

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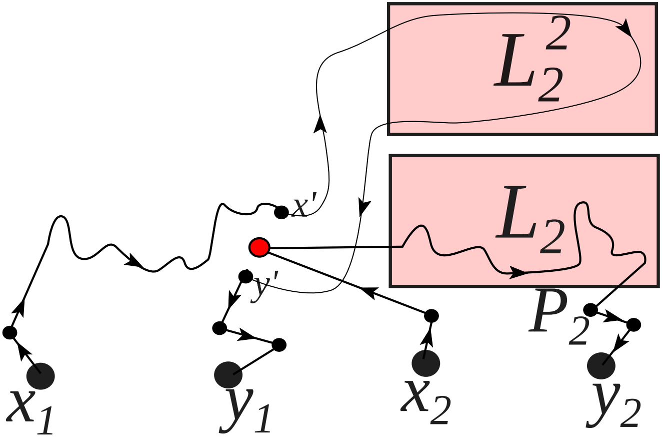

Other applications of linkage structures

Conjecture (Thomassen)

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There is a constant C such that every strongly $Ck^2(\log k)^2$-connected tournament contains k edge-disjoint Hamiltonian cycles.

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Theorem (P.)

There is a constant C such that every strongly Ck^2-connected tournament contains k edge-disjoint Hamiltonian cycles.

Other applications of linkage structures

Theorem (Kühn, Osthus, and Townsend)

There is a function $f(k,t)$ every strongly $f(k,t)$-connected tournament can be partitioned into t strongly k-connected subtournaments.

This was conjectured by Thomassen.

Theorem (Kühn, Osthus, and Townsend)

There is a function $g(k)$ with the following property. For any natural numbers $n_1, \ldots, n_k$ satisfying $\sum_{i=1}^{k} n_i = n$, the vertices of every strongly $g(k)$-connected tournament T on n vertices can be partitioned into cycles $C_1, \ldots, C_k$ such that $|C_i| = n_i$.

Song conjectured that $g(k) = k$.

Open problems

Conjecture (Kühn, Osthus, and Townsend)

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Conjecture (Song)

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Problem

Every $22k$-connected tournament is k-linked.