Nonnegative $k$-sums in a set of numbers

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Nonnegative sums

\{5, 3, -6, -1\}. Nonnegative sums:
\{5, 3, -6\}, \{5, 3, -1\}, \{5, 3\}, \{5, -1\}, \{3, -1\}, \{5\}, \{3\}.

Problem

Let \(x_1, \ldots, x_n\) be a set of numbers satisfying \(x_1 + x_2 + \cdots + x_n \geq 0\).
How few subsets can have nonnegative sum?
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Problem

Let \(x_1, \ldots, x_n\) be a set of numbers satisfying \(x_1 + x_2 + \cdots + x_n \geq 0\). How few subsets of order \(k\) can have nonnegative sum?
Conjecture (Manickam, Miklós, Singhi)

Let $n \geq 4k$ and $x_1, \ldots, x_n$ be a set of numbers satisfying $x_1 + x_2 + \cdots + x_n \geq 0$. At least $\binom{n-1}{k-1}$ subsets of order $k$ have nonnegative sum.
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The bound is seen to be best possible by again choosing \( x_1 = n \), and \( x_2 = x_3 = \cdots = \ldots x_n = -1 \).
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- The bound is seen to be best possible by again choosing $x_1 = n$, and $x_2 = x_3 = \cdots = x_n = -1$.
- “$n \geq 4k$” is motivated by a construction at $n = 3k + 1$ ($x_1 = x_2 = x_3 = 2 - 3k$ and $x_4 = \cdots = x_{3k+1} = 3$).
Known results

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- True when \( n \equiv 0 \pmod{k} \). (Manickam and Singhi)
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- True when \( n \equiv 0 \pmod{k} \). (Manickam and Singhi)
- True when \( n \) is large compared to \( k \):
  - \( n \geq k (4e \log k) k \) (Tyomkyn).
  - \( n \geq 33k^2 \) or \( n \geq 2k^3 \) (Alon, Huang, and Sudakov).
  - \( n \geq 2k^3 \) (Aydinian and Blinovsky).
  - \( n \geq 3k^3/2 \) (Frankl).
  - \( n \geq 8k^2 \) (Chowdhury, Sarkis, Shahriari).
  - \( n \geq 10^{46} k \) (P.).
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  - \( n \geq (k - 1)(k^k + k^2) + k \) (Manickam and Miklós).
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Katona’s cycle method

How to prove MMS conjecture:

We have set $\{x_1, \ldots, x_k\}$ with $x_1 + \cdots + x_k \geq 0$.

Choose a random cyclic ordering of $x_1, \ldots, x_n$.

Count $E = E(\text{number of nonnegative } k\text{-intervals})$ in two different ways.

$E = (\text{number of nonnegative } k\text{-sets})$

$P(k\text{-set forms an interval}) = (\text{number of nonnegative } k\text{-sets})$

$k \choose n - k \frac{n!}{n! n!}.$

Lemma (easy)

Any nonnegative weighting of $\mathbb{Z}_n$ has at least $k$ nonnegative $k$-intervals.

Therefore $E \geq k$ and so $\text{number of nonnegative } k\text{-sets} \geq \binom{n - 1}{k - 1}$.
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\[ E = (\text{number of nonneg. k-sets}) \]
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\[ = k! \left( \frac{n - k}{n} \right)! \]

Lemma (easy)

Any nonneg. weighting of \( \mathbb{Z}_n \), has at least \( k \) nonneg. k-intervals

Therefore \( (\text{number of nonneg. k-sets}) \geq k \) and so

\[ \text{number of nonneg. k-sets} \geq \binom{n - 1}{k - 1} \]
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E = (\text{number of nonneg. } k\text{-sets}) \mathbb{P}(\text{k-set forms an interval})
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Lemma (easy)

Any nonneg. weighting of \( \mathbb{Z}_n \), has at least \( k \) nonneg. \( k \)-intervals

- Therefore (number of nonneg. \( k \)-sets) \( \frac{k!(n-k)!}{n!} n \geq k \) and so

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\text{number of nonneg. } k\text{-sets} \geq \binom{n-1}{k-1}.
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Lemma (mostly false)

Any nonneg. weighting of \( \mathbb{Z}_n \), has at least \( k \) nonneg. k-intervals

- Therefore \( (\text{number of nonneg. k-sets}) \cdot \frac{k!(n-k)!}{n!} \cdot \frac{1}{n} \cdot n \geq k \) and so

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\text{number of nonneg. k-sets} \geq \binom{n-1}{k-1}.
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Why is the easy lemma is false?

$k = 3$, $n = 16$. Here is a weighting of the cycle with only one nonnegative interval:

![Diagram of a cycle with weights and a highlighted nonnegative interval.]
Katona’s cycle method

How to prove MMS conjecture when $n \equiv 0 \pmod{k}$

We have set $\{x_1, ..., x_k\}$ with $x_1 + \cdots + x_k \geq 0$.

Choose a random ordering of $x_1, ..., x_n$.

Count $E = E(\text{number of nonnegative intervals of the form} \{t_k, \ldots, t_k + k - 1\})$ in two different ways.

$E = (\text{number of nonnegative } k\text{-sets}) P(k\text{-set forms an interval})$

$= (\text{number of nonnegative } k\text{-sets}) \frac{k!}{(n-k)!n!}$

Lemma (really easy)
For any nonnegative weighting of $\mathbb{Z}_n$, there is at least 1 nonnegative $k$-interval of the above form.

Therefore $\text{number of nonnegative } k\text{-sets} \geq \left(\frac{n-k}{k-1}\right)$.
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\text{Count } E = E(\text{number of nonnegative intervals of the form } \{t_k, \ldots, t_k+k-1\}) \text{ in two different ways.}
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E = (\text{number of nonneg. } k\text{-sets}) \cdot P(k\text{-set forms an interval}) = (\text{number of nonneg. } k\text{-sets}) \cdot \frac{k!}{(n-k)! n! n^k}
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Lemma (really easy)

For any nonnegative weighting of $\mathbb{Z}_n$, there is at least 1 nonnegative $k$-interval of the above form.

Therefore $(\text{number of nonneg. } k\text{-sets}) \cdot \frac{k!}{(n-k)! n! n^k} \geq 1$ and so $(\text{number of nonneg. } k\text{-sets}) \geq \left(\frac{n-k}{n-1}\right)$. 

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**Lemma (really easy)**

*For any nonnegative weighting of \( \mathbb{Z}_n \), there is at least 1 nonnegative k-intervals of the above form.*

- Therefore \( (\text{number of nonneg. } k\text{-sets}) \frac{k!(n-k)!}{n!} \frac{n}{k} \geq 1 \) and so

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\text{number of nonneg. } k\text{-sets} \geq \binom{n-1}{k-1}.
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A $k$-uniform hypergraph $\mathcal{H}$ has the **MMS property** if for any weighting of $V(\mathcal{H})$, $w : V(\mathcal{H}) \to \mathbb{R}$, satisfying $\sum_{v \in \mathcal{H}} w(v) \geq 0$, there are at least $\delta(\mathcal{H})$ nonnegative edges in $\mathcal{H}$.
MMS property

Definition

A $k$-uniform hypergraph $\mathcal{H}$ has the **MMS property** if for any weighting of $V(\mathcal{H})$, $w : V(\mathcal{H}) \to \mathbb{R}$, satisfying $\sum_{v \in \mathcal{H}} w(v) \geq 0$, there are at least $\delta(\mathcal{H})$ nonnegative edges in $\mathcal{H}$.

The averaging argument from the last few slides shows that:

Lemma

*Suppose that there is a regular $k$-uniform hypergraph on $n$ vertices with the MMS property.*

*Then the Manickam-Miklós-Singhi Conjecture holds for that $n$ and $k$.*
Theorem (P.)

For \( n \geq 10^{46} k \), there are \( k(k - 1)^2 \)-regular \( k \)-uniform hypergraphs \( \mathcal{H}_{n,k} \) on \( n \) vertices with the MMS property.
MMS property

**Theorem (P.)**

For $n \geq 10^{46} k$, there are $k(k - 1)^2$-regular $k$-uniform hypergraphs $\mathcal{H}_{n,k}$ on $n$ vertices with the MMS property.

Vertices of $\mathcal{H}_{n,k}$ are $\mathbb{Z}_n$. 

Edges of $\mathcal{H}_{n,k}$ are double intervals where the distance between the intervals is less than $k$. 

i.e. sets of the form $[x, x + i - 1] \cup [x + i + j, x + k + j - 1]$ for $i, j < k$. 
Sketch of proof

Suppose that $\mathcal{H}_{n,k}$ contains less than $d(\mathcal{H}_{n,k})$ nonnegative edges. We prove two claims:

Claim 1 Let $I$ be an interval in $V(\mathcal{H}_{n,k})$ of length $\leq n - 2k$. Then there is a negative interval $J$ of order at most $|I| + 2k$ containing $I$.

Claim 2 Let $I$ be an interval in $V(\mathcal{H}_{n,k})$ with $|I| \geq 20k$ containing no nonnegative edges. Then $I$ is negative.

If $n \geq 30k^4$, by the Pigeonhole Principle there is an interval $I$ of length $30k$ containing no nonnegative edges.
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Open problems

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What hypergraphs have the MMS property?
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Theorem (Huang and Sudakov)

For $n \geq 10k^3$, every hypergraph with all codegrees equal has the MMS property.
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What is the complexity of deciding whether a hypergraph has the MMS property?

Problem

Characterize all (2-uniform) graphs with the MMS property.