Nonnegative k-sums in a set of numbers

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Problem

Let $x_1, \dots, x_n$ be a set of numbers satisfying $x_1 + x_2 + \cdots + x_n \geq 0$. How few subsets of order k can have nonnegative sum?

Nonnegative sums

Conjecture (Manickam, Miklós, Singhi)

Let $n \geq 4k$ *and* $x_1, \ldots, x_n$ *be a set of numbers satisfying* $x_1 + x_2 + \cdots + x_n \geq 0$. *At least* $\binom{n-1}{k-1}$ *subsets of order* k *have nonnegative sum.*

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- The bound is seen to be best possible by again choosing $x_1 = n$, and $x_2 = x_3 = \cdots = \ldots x_n = -1$.
- "$n \geq 4k$" is motivated by a construction at $n = 3k + 1$ ($x_1 = x_2 = x_3 = 2 - 3k$ and $x_4 = \cdots = x_{3k+1} = 3$).

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  - $n \geq 10^{46}k$ (P.)

Katona's cycle method

How to prove MMS conjecture:

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Lemma (mostly false)

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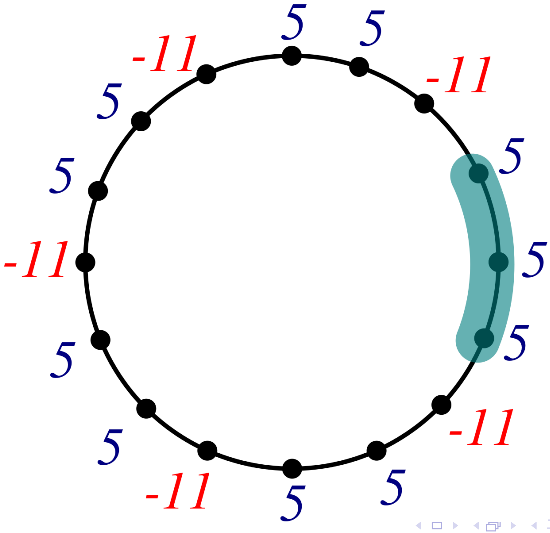
Why is the easy lemma is false?

$k = 3, n = 16$. Here is a weighting of the cycle with only one nonnegative interval:



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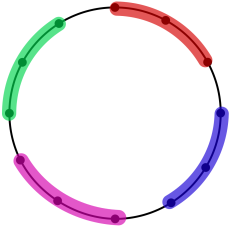

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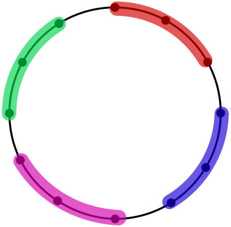

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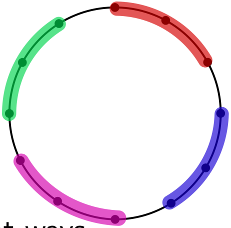

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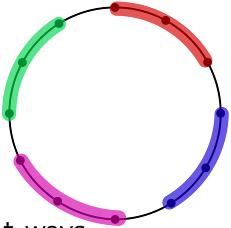

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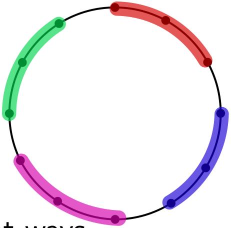

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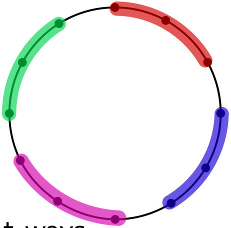

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MMS property

Definition

A k-uniform hypergraph $\mathcal{H}$ has the **MMS property** if for any weighting of $V(\mathcal{H})$, $w : V(\mathcal{H}) \to \mathbb{R}$, satisfying $\sum_{v \in \mathcal{H}} w(v) \geq 0$, there are at least $\delta(\mathcal{H})$ nonnegative edges in $\mathcal{H}$.

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The averaging argument from the last few slides shows that:

Lemma

Suppose that there is a regular k-uniform hypergraph on n vertices with the MMS property.
Then the Manickam-Miklós-Singhi Conjecture holds for that n and k.

MMS property

Theorem (P.)

For $n \geq 10^{46}k$, there are $k(k-1)^2$-regular k-uniform hypergraphs $\mathcal{H}_{n,k}$ on n vertices with the MMS property.

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Vertices of $\mathcal{H}_{n,k}$ are $\mathbb{Z}_n$.
Edges of $\mathcal{H}_{n,k}$ are *double intervals* where the distance between the intervals is less than k.
i.e. sets of the form $[x, x+i-1] \cup [x+i+j, x+k+j-1]$ for $i, j < k$.

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Let I be an interval in $V(\mathcal{H}_{n,k})$ with $|I| \geq 20k$ containing no nonnegative edges. Then I is negative.

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If $n \geq 30k^4$, by the Pigeonhole Principle there is an interval I of length $30k$ containing no nonnegative edges.

Open problems

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What hypergraphs have the MMS property?

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Problem

Characterize all (2-uniform) **graphs** *with the MMS property.*