Calculating Ramsey numbers by partitioning coloured graphs

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Ramsey Theory

The Ramsey Number $R(G, H)$ is the smallest n for which any 2-edge-colouring of K_n contains either a red G or a blue H.

Theorem (Ramsey, 1930)

$R(K_n, K_n)$ is finite for every n.

The following bounds hold

$$\sqrt{2}^n \leq R(K_n, K_n) \leq 4^n.$$

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$$R(P_n, K_m) = (n-1)(m-1) + 1.$$

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The following lower bound holds for all G and H (Chvatal; Harary and Burr).

$$R(G, H) \geq (\chi(H) - 1)(|G| - 1) + \sigma(H)$$

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Proof.

$\square$

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The vertices of every graph G can be covered by $\alpha(G)$ disjoint cycles.

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Theorem (Gallai-Milgram, 1960)

The vertices of every directed graph D can be covered by $\alpha(D)$ disjoint directed paths.

Ramsey Theory

Theorem (Gerencsér and Gyárfás, 1966)

For $n \geq m$*,*

$$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

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Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

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THEOREM 1. *For $k \geqq l$ we have*

$$g(k,l) = k + \left[\frac{l+1}{2}\right]. \tag{1}$$

Considering the other special case of this type of problems, let $f_r(n)$ denote the greatest integer with the property, that colouring the edges of a complete n-tuple g with r colours arbitrarily, there exists always a one-coloured connected subgraph with at least $f_r(n)$ vertices.

It is easy to see the following remark of P. ERDŐS: if a graph is not connected then its complement is connected, i.e. $f_2(n) = n$. We shall prove

THEOREM 2.

$$f_3(n) = \left[\frac{n+1}{2}\right]. \tag{2}$$

Now we turn to the proof of Theorem 1. First we prove $g(k,l) \leqq k + \left[\frac{l+1}{2}\right]$ by induction on k. For $k=1$ the Theorem evidently holds and let us suppose that for all k-s less than this the statement is true. Let us consider a graph G with $k+\left[\frac{l+1}{2}\right]$ vertices. If $l<k$, then for any subgraph of G with $k-1+\left[\frac{l+1}{2}\right]$ points holds that either itself contains a path of length $k-1$, or its complement a path of length l. For $l=k$ we consider a subgraph with $k-1+\left[\frac{l}{2}\right]$ points. This or its complement contains a path of length $k-1$. Thus in every case can be supposed, that the length of the longest path of G is $k-1$. Let $U_1, U_2, \ldots, U_k$ be the consecutive vertices of such a path and $U = \{U_1, \ldots, U_k\}$. We denote the remaining vertices by $V_1, \ldots, V_{\left[\frac{l+1}{2}\right]}$ and the set of them by $V = \{V_1, \ldots, V_{\left[\frac{l+1}{2}\right]}\}$.

It clearly holds that

(i) for all $V_i \in V$ either $V_iU_j \in \bar{G}$ or $V_iU_{j+1} \in \bar{G}$

(ii) for all $V_i \in V$ $V_iU_1 \in \bar{G}$ and $V_iU_k \in \bar{G}$

(iii) for $V_{i1}, V_{i2}, V_{i3} \in V$ and $U_j, U_{j+1} \in U$

at least one of the latest points is connected in $\bar{G}$ with at least two of V_{i1}, V_{i2}, V_{i3}.

Consider a maximal path of $\bar{G}$ not containing U_1, U_k with the property that any edge of it connects a point of U with a point of V, and its endpoints are in V; let us denote the endpoints by A and B, and the path by S. If S contains all points of V, then by adding the edges U_1A, BU_k we have a path of length $2\left[\frac{l+1}{2}\right] \geqq l$ in $\bar{G}$. So we may suppose that the set of points V not contained by S is not empty. Let this set be called W. Consider a maximal path q of $\bar{G}$ not containing U_1, U_k and having no common points with S, such that any edge of it connects a point of U with a point of W and the endpoints of it,

called by C and D, are in W. We show that all points of V are contained either in S or in q. Suppose that $X \in V$ but $X \notin S$, $X \notin q$. It is clear, that the number of vertices of S and q in U is at most $\left[\frac{l+1}{2}\right] - 3 < \left[\frac{k-3}{2}\right] = \left[\frac{k-2-1}{2}\right]$ since $l \leqq k$. So there exist two points $U_i, U_{i+1} \in \{U_2, \ldots, U_{k-1}\}$ which do not belong either to S or to q. Applying (iii) for $A, C, X \in V$ and $U_i, U_{i+1} \in U$ we have a contradiction to the maximal properties of S and q.

So the sum of the length of S and q is $2\left[\frac{l+1}{2}\right]-4$. We add them the edges U_1A, BU_k, U_kC, DU_1 and so we have a circuit of length $2\left[\frac{l+1}{2}\right]$ in $\bar{G}$. For odd l this contains a desired path with length l. For even l an easy reasoning shows that there are $U_i, U_{i+1} \in U$ which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length l in $\bar{G}$. That completes the proof.

Now we give examples for graphs G with $k+\left[\frac{l+1}{2}\right]-1$ points that have no path of length k, and for them at the same time $\bar{G}$ have no path of length l.

a) Let G consist of the disjoint graphs H_1, H_2 with k and $\left[\frac{l+1}{2}\right]-1$ points respectively, where the graph H_1 is complete.

b) For even l we can leave one of the edges of H_1. These graphs possess obviously the desired property.[1]

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph G into three classes, i.e. let the edges of G be coloured with red, yellow and blue colours. So we get the graphs G_r, G_y and G_b formed by the red, yellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of G_r. Let us take a maximal red-connected subgraph R. It may be supposed that R is not empty and $\pi(R) < \pi(G) = n$. Let B be a point of G such that $B \notin R$. Since R is a maximal connected subgraph of G_r, BR_i is not red for $R_i \in R$. So one may suppose that there are at least $\frac{1}{2}\pi(R)$ points of R which are connected with B by blue edges. Let V denote the set of these points of R and W be the maximal blue-connected subgraph that contains B. If Y is a point such that $Y \notin R$ and $Y \notin W$ then YV_i is yellow for $V_i \in V$. Let Q denote the maximal yellow-connected subgraph that contains Y. If there is no such Y, Q denotes the empty set. R, W, Q contain together all points of G. Namely any points $S \notin R$ is connected with a

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Theorem (Gyárfás and Lehel; Faudree and Schelp, 1973)

$$R_{K_{n,n}}(P_n, P_m) \approx n + m$$

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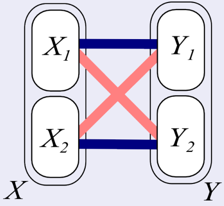
Theorem (Gyárfás and Lehel, 1973)

Let G be a 2-edge-coloured balanced complete bipartite graph. Then one of the following holds.

- *G looks like this:* 

- *Then there are two disjoint monochromatic paths covering all, except possibly one vertex in G.*

Partitioning coloured graphs

Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

Conjecture (Gyárfás, 1989)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic paths.

This theorem and conjecture gave rise to a number of results.

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This conjecture led to...

- Every r-edge-coloured infinite complete graph can be covered by r infinite monochromatic paths. [Rado, 1987]
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This conjecture led to...

- Every 3-edge-coloured K_n has 3 monochromatic cycles covering $n - o(n)$ vertices. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2011]
- Not every 3-edge-coloured K_n can be covered by 3 disjoint monochromatic cycles. [P., 2013]
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- Suppose that we have a sequence $\mathcal{G} = \{G_0, G_1, G_2, \dots\}$ of graphs with maximum degree $\leq \Delta$. Every 2-edge-coloured complete graph can be covered by at most $2^{C\Delta \log \Delta}$ monochromatic copies of graphs from $\mathcal{G}$. [Grinshpun and Sárközy, 2013]

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Results

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Suppose that the edges of K_n are 2-coloured. Then K_n can be covered by k disjoint red paths and a disjoint blue balanced complete $(k+1)$-partite graph.

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Applications

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Suppose that the edges of K_n are 2-coloured. Then K_n can be covered by k disjoint red paths and a disjoint blue balanced complete $(k+1)$-partite graph.

- Generalises original Gerencsér-Gyárfás path partitioning theorem.

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- Generalises original Gerencsér-Gyárfás path partitioning theorem.
- Can be used to prove the $r = 3$ case of Gyárfás Conjecture.

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- $R(P_n, K_i^t) = (t-1)(n-1) + t(i-1) + 1$
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- Might be useful for finding $R(P_n, H)$ for other graphs H..?

Proof

Theorem

Every 2-edge-coloured complete graph can be covered by a red path and a disjoint blue balanced complete bipartite graph.

Proof.

□

Open problems

Conjecture

Every 2-edge-coloured complete tripartite graph can be covered by two disjoint monochromatic paths.

Conjecture (Gyárfás and Sarközy)

Every complete r-uniform hypergraph H can be covered by $\alpha(H)$ disjoint loose cycles.

Problem

Every r-edge-coloured complete graph can be covered by $1000r$ monochromatic paths.

Open problems

Problem

Prove natural statements of the form "Every 2-edge-coloured complete graph can be covered by a red graph G and a disjoint blue graph H with G and H having particular structures".

Known results of this type:

- G and H paths [Gerencsér and Gyárfás].
- G and H cycles [Łuczak, Rödl, and Szemerédi; Allen; Bessy and Thomassé].
- G a matching, H a complete graph [folklore].
- G a forest of k paths, H a balanced complete $(k+1)$-partite graph. [P.]
- G a cycle, H a graph with $\Delta(H) \geq \frac{1}{2}(|H|-1)$. [P.]