# Calculating Ramsey numbers by partitioning coloured graphs

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The Ramsey Number R(G, H) is the smallest *n* for which any 2-edge-colouring of  $K_n$  contains either a red *G* or a blue *H*.

Theorem (Ramsey, 1930)

 $R(K_n, K_n)$  is finite for every n.

The following bounds hold

$$\sqrt{2}^n \leq R(K_n, K_n) \leq 4^n.$$

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Theorem (Erdős, 1947)

$$R(P_n, K_m) = (n-1)(m-1) + 1.$$

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The following lower bound holds for all G and H (Chvatal; Harary and Burr).

$$R(G,H) \geq (\chi(H)-1)(|G|-1) + \sigma(H)$$

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Ramsey Theory Theorem (Erdős, 1947)  $R(P_n, K_m) = (n-1)(m-1) + 1.$ 

Proof.

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# Theorem (Gerencsér and Gyárfás, 1966) For $n \ge m$ , $R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$

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Theorem (Gerencsér and Gyárfás, 1966) For  $n \ge m$ ,  $R(P_n, P_m) = n + \left|\frac{m}{2}\right| - 1.$ 

#### Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

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THEOREM 1. For  $k \ge l$  we have

(1)

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$$g(k,l) = k + \left[\frac{l+1}{2}\right]$$

Considering the other special case of this type of problems, let  $f_n(n)$  denote the greatest integer with the property, that colouring the edges of a complete *n*-tuple g with *r* colours arbitrarily, there exists always a one-coloured connected subgraph with at least  $f_n(n)$  vertices.

It is easy to see the following remark of P. ERDős: if a graph is not connected then its complement is connected, i.e.  $f_2(n) = n$ . We shall prove

THEOREM 2.

 $f_{\mathfrak{z}}(n) = \left[\frac{n+1}{2}\right]$ 

Now we turn to the proof of Theorem 1. First we prove  $g(k, l) \leq k + \frac{l+1}{2}$ 

by induction on k. For k=1 the Theorem evidently holds and let us suppose that for all ks less than this the statement is true. Let us consider a graph G with  $k + \left\lfloor \frac{l+1}{2} \right\rfloor$  vertices. If l - k, then for any subgraph of G with  $k - 1 + \left\lfloor \frac{l+1}{2} \right\rfloor$  points holds that either itself contains a path of length k - 1, or its complement a path of length l. For l - k we consider a subgraph with  $k - 1 + \left\lfloor \frac{l}{2} \right\rfloor$  points.

This or its complement contains a path of length k-1. Thus in every case can be supposed, that the length of the longest path of G is k-1. Let  $U_1, U_2, \ldots, U_k$  be the consecutive vertices of such a path and  $U = \{U_1, \ldots, U_k\}$ . We denote the remaining vertices by  $V_1, \ldots, V_k$  field.

$$= \left\{ V_1, \ldots, V_{\left[\frac{l+1}{2}\right]} \right\}.$$

It clearly holds that

(i) for all V<sub>i</sub> ∈ V either V<sub>i</sub>U<sub>j</sub> ∈ G or V<sub>i</sub>U<sub>j+1</sub> ∈ G
(ii) for all V<sub>i</sub> ∈ V V<sub>i</sub>U<sub>1</sub> ∈ G and V<sub>i</sub>U<sub>k</sub> ∈ G
(iii) for V<sub>i</sub>, V<sub>i</sub>, V<sub>i</sub> ∈ V and U<sub>p</sub>, U<sub>j+1</sub> ∈ U

at least one of the latest points is connected in G with at least two of Vn, Vn, Vn,

Consider a maximal path of  $\overline{Q}$  not containing  $U_{\nu}$ ,  $U_{\nu}$  with the property that any edge of it connects a point of U with a point of V, and its endpointsare in V; let us denote the endpoints by A and B, and the path by S. If S contains all points of V, then by adding the edge  $U_A, BU_{\nu}$  we have a path of length  $2 \left\lfloor \frac{I-1}{2} \right\rfloor_{\nu} i$  in  $\overline{G}$ . So we may suppose that the set of points V not contained by  $\overline{Z}$  in U endpoints D with use the set be called W. Consider a maximal path of  $\overline{G}$  out containing  $U_{\nu}, U_{\nu}$  and having no common points with S, such that  $\overline{V}$  edge of it connects a point of U with a point of W and the endpoints of it, called by C and D, are in W. We show that all points of V are contained either in S or in a Suppose that  $X \in V$  but  $X \in S$ ,  $X \notin q$ . It is clear, that the number of vertices of S and q in U is at most  $\left| \frac{k-1}{1} \right| -3 - \left| \frac{k-3}{2} \right| = \left| \frac{k-2-1}{2} \right|$  since it is the point  $| \frac{k-1}{1} \right| -3 - \left| \frac{k-3}{2} \right| = \left| \frac{k-2}{2} \right|$  since it is point  $2 + \left| \frac{k-3}{1} \right| -3 - \left| \frac{k-3}{2} \right| = \left| \frac{k-2}{2} \right|$  since  $| \frac{k-2}{2} \right|$  since  $| \frac{k-2}{2} \right| = \left| \frac{k-2}{2} \right|$  since  $| \frac{k-2}{2} \right| = \left| \frac{k-2}{2} \right|$  since  $| \frac{k-2}{2}$  since  $| \frac{k-2}{2} \right|$  since  $| \frac{k-2}{2}$  sin

So the sum of the length of S and q is  $2\left[\frac{l+1}{2}\right] - 4$ . We add them the edges

 $U_1A$ ,  $BU_{k}$ ,  $U_kC$ ,  $DU_1$  and so we have a circuit of length  $2\left[\frac{l+1}{2}\right]$  in  $\overline{G}$ . For odd *l* this contains a desired path with length *l*. For even *l* an easy reasoning

Show that there are  $U_n e_{i+1} \in U$  which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length 1 in  $\overline{G}$ . That completes the proof.

Now we give examples for graphs G with  $k + \left[\frac{l+1}{2}\right] - 1$  points that have

no path of length k, and for them at the same time  $\widetilde{G}$  have no path of length l.

a) Let G consist of the disjoint graphs  $H_1$ ,  $H_2$  with k and  $\left\lfloor \frac{l+1}{2} \right\rfloor - 1$  points

respectively, where the graph  $H_1$  is complete.

b) For even l we can leave one of the edges of  $H_1$ . These graphs possess obviously the desired property.<sup>1</sup>

Now we turn to the proof of Theorem 2. We consider a classification of the edge of a complete graph G into three classes, i.e. let the edges of be coloured with red, yellow and blue colours. So we get the graphs G, G, and G<sub>3</sub> formed by the red, velow and blue degraph is for example red-connected altric is a connected subgraph of G. Let us take a maximal red-connected subgraph A. It may be supposed that R is not empty and  $\pi(R) \approx -nected subgraph of G$ , R = 0, R

there are at least  $\frac{1}{2}\pi_{x}R$ ) points of R which are connected with B by blue edges.

Let V denote the set of these points of R and W be the maximal blue-connected subgraph that contains B. If Y is a point such that Y  $\notin R$  and Y  $\notin W$  then YV<sub>t</sub> is yellow for V<sub>t</sub>  $\in V$ . Let Q denote the maximal yellow-connected subgraph that contains Y. If there is no such Y, Q denotes the empty set R, W, Q contain together all points of G. Namely any points S  $\in R$  is connected with a

<sup>1</sup> The weaker result  $g(k, l) \equiv k+l$  can be easily proved. Let us consider any vertex P and a pair of paths of G and  $\overline{G}$  without common vertices excet P. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all P and all pairs). From that the statement follows.

#### L. GERENCSÉR AND A. GYÁRFÁS

THEOREM 1. For  $k \ge l$  we have

(1)

 $g(k, l) = k + \left[\frac{l+1}{2}\right].$ Considering the other special case of this type of problems, let  $f_n(n)$  denote the greatest integer with the property, that colouring the edges of a complete n-tuple g with r colours arbitrarily, there exists always a one-coloured connected subgraph with at least  $f_i(n)$  vertices.

It is easy to see the following remark of P. ERDOS: if a graph is not connected then its complement is connected, i.e.  $f_0(n) = n$ . We shall prove

THEOREM 2.

 $f_3(n) = \left[\frac{n+1}{2}\right]$ 

Now we turn to the proof of Theorem 1. First we prove  $g(k, l) \equiv k + l$ 

by induction on k. For k = 1 the Theorem evidently holds and let us suppose that for all k-s less than this the statement is true. Let us consider a graph G with  $k + \left\lfloor \frac{l+1}{r} \right\rfloor$  vertices. If l < k, then for any subgraph of G with k - 1 + 1points holds that either itself contains a path of length k-1, or its complement a path of length l. For l=k we consider a subgraph with k-1+points.

This or its complement contains a path of length k-1. Thus in every case can be supposed, that the length of the longest path of G is k-1. Let  $U_1, U_2, \ldots, U_k$ be the consecutive vertices of such a path and  $U = \{U_1, \ldots, U_k\}$ . We denote the remaining vertices by  $V_1, \ldots, V_{\lceil l+1 \rceil}$  and the set of them by V =

$$= \left\{ V_1, \ldots, V_{\left[\frac{l+1}{2}\right]} \right\}.$$

It clearly holds that

(i) for all  $V_i \in V$  either  $V_i U_i \in \overline{G}$  or  $V_i U_{i+1} \in \overline{G}$ (ii) for all  $V_i \in V$   $V_i U_i \in \overline{G}$  and  $V_i U_k \in \overline{G}$ (iii) for  $V_{i1}$ ,  $V_{i2}$ ,  $V_{i3} \in V$  and  $U_{i1}$ ,  $U_{i+1} \in U$ 

at least one of the latest points is connected in G with at least two of Vn, Vm, Vm,

Consider a maximal path of  $\overline{G}$  not containing  $U_1, U_k$  with the property that any edge of it connects a point of U with a point of V, and its endpointsare in V: let us denote the endpoints by A and B, and the path by S. If S con tains all points of V, then by adding the edges  $U_1A$ ,  $BU_k$  we have a path of length  $2\left\lfloor \frac{l+1}{2}\right\rfloor \ge l$  in  $\overline{G}$ . So we may suppose that the set of points V not contained by S is not empty. Let this set be called W. Consider a maximal path g of  $\overline{G}$  not containing  $U_1, U_2$  and having no common points with S, such that any edge of it connects a point of U with a point of W and the endpoints of it, called by C and D, are in W. We show that all points of V are contained either in S or in q. Suppose that  $X \in V$  but  $X \notin S$ ,  $X \notin q$ . It is clear, that the number of vertices of S and q in U is at most  $\left\lceil \frac{l+1}{2} \right\rceil - 3 < \left\lceil \frac{k-3}{2} \right\rceil = \left\lfloor \frac{k-2-1}{2} \right\rceil$  $l \leq k$ . So there exist two points  $U_i, U_{i+1} \in \{U_2, \dots, U_{k-1}\}$  which do not belong either to S or to q. Applying (iii) for A, C, X \in V and  $U_i, U_{i+1} \in U$  we have a contradiction to the maximal properties of S and q.

So the sum of the length of S and q is  $2\left\lfloor \frac{l+1}{2} \right\rfloor - 4$ . We add them the edges

 $U_1A$ ,  $BU_k$ ,  $U_kC$ ,  $DU_1$  and so we have a circuit of length  $2\left[\frac{l+1}{c}\right]$  in  $\overline{G}$ . For odd l this contains a desired path with length l. For even l an easy reasoning

shows that there are  $U_i, U_{i+1} \in U$  which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length l in  $\overline{G}$ . That completes the proof.

Now we give examples for graphs G with  $k + \left\lfloor \frac{l+1}{2} \right\rfloor - 1$  points that have

no path of length k, and for them at the same time  $\tilde{G}$  have no path of length l.

a) Let G consist of the disjoint graphs  $H_1$ ,  $H_2$  with k and  $\left\lfloor \frac{l+1}{2} \right\rfloor - 1$  points

respectively, where the graph H, is complete.

b) For even l we can leave one of the edges of  $H_1$ . These graphs possess obviously the desired property.1

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph G into three classes, i.e. let the edges of G be coloured with red, yellow and blue colours. So we get the graphs Gr, Gy and Gh formed by the red. vellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of G., Let us take a maximal red-connected subgraph R. It may be supposed that R is not empty and  $\pi(R) <$  $\prec \pi(G) = n$ . Let B be a point of G such that  $B \notin R$ . Since R is a maximal connected subgraph of G.,  $BR_i$  is not red for  $R_i \in R$ . So one may suppose that

there are at least  $\frac{1}{n} \pi(R)$  points of R which are connected with B by blue edges.

Let V denote the set of these points of R and W be the maximal blue-connected subgraph that contains B. If Y is a point such that  $Y \in R$  and  $Y \in W$  then  $YV_i$  is yellow for  $V_i \in V$ . Let Q denote the maximal yellow-connected subgraph that contains Y. If there is no such Y. O denotes the empty set, R. W. O contain together all points of G. Namely any points  $S \notin R$  is connected with a

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Considering the other special case of this type of problems, let  $f_{*}(n)$  denote the greatest integer with the property, that colouring the edges of a complete *n*-tuple g with *r* colours arbitrarily, there exists always a one-coloured connected subgraph with at least  $f_{*}(n)$  vertices.

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THEOREM

(2)  $f_3(n) =$ 

Now we turn to the proof of Theorem 1. First we prove  $g(k, l) \equiv k + \left\lfloor \frac{l+1}{2} \right\rfloor$ 

by induction on k. For k=1 the Theorem evidently holds and let us suppose the

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called by C and D, are in W. We show that all points of V are contained either in S or in q, suppose that  $X \in V$  but  $X \in S, X \in q$ . It is clear, that the number of vertices of S and q in U is at most  $\left\lfloor \frac{1}{2} \right\rfloor - 1 - \left\lfloor \frac{R}{2} \right\rfloor = 1 = \left\lfloor \frac{R}{2} \right\rfloor = 1$ is R. So there exist two points U,  $U_{1,rr} \in \{U_{pr}, \cdots, U_{n-1}\}$  which do not belong either to S or to q, applying (ii) for  $A, C, X \in V$  and  $U_{1,rs} \in U$  we have a contradiction to the maximal properties of S and q.

So the sum of the length of S and q is  $2\lfloor \frac{l+1}{2} \rfloor -4$ . We add them the edges

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It clearly holds that

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Consider a maximal path of  $\overline{d}$  not containing  $U_{b}, U_{a}$  with the property that any edge of it connects a point of U with a point of V, and its endpointsare in V; let us denote the endpoints by A and B, and the path by S. If S contains all points of V, then by adding the edges  $U_{i}A$ ,  $BU_{i}$  we have a path of length  $2 \left\lfloor \frac{1}{2} + 1 \right\rfloor_{a} i$  in  $\overline{G}$ , so we may suppose that the set of points V not contained by  $\overline{C}$  are anoth I at this each bound W. Consider a recursion and

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<sup>1</sup> The weaker result g(k,l) = k+l can be easily proved. Let us consider any vertex  $l^p$  and a pair of paths of d and  $\overline{d}$  without common vertices except P. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all P and all pairs.) From that the statement follows.

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L. GERENCSÉR AND A. GYÁRPÁS

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It clearly holds that

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Consider a maximal path of  $\overline{a}$  not containing  $U_{\nu}$ ,  $U_{\nu}$  with the property that any edge of it connects a point of U with a point of V, and its endpointsare in V; let us denote the endpoints by A and B, and the path by S. If S contains all points of V, then ty adding the edge  $UA, BU_{\nu}$  we have a path of length  $2\left\lfloor \frac{1}{2} \right\rfloor_{\mu} B \mid in \overline{B}$ . So we may suppose that the set of points V not contained by S in not entry, Let this set be called V. Consider a maximal path

q of  $\overline{G}$  not containing  $U_1, U_k$  and having no common points with S, such that any edge of it connects a point of U with a point of W and the endpoints of it,

toget on a somprive graphy of into time classes, i.e. we use togets of o we consider with red, yellow and blue colours. So we get the graphe  $G_0$ ,  $G_0$  and  $G_0$  formed evaluation of the state  $G_0$  and  $G_0$  formed evaluation of the state  $G_0$  and  $G_0$  and  $G_0$  are state and  $G_0$  and  $G_0$  and  $G_0$  are state an aximal red-connected subscraph  $B_0$ . If any he supposed that  $R_0$  is not entry of  $\pi_0$ , if at maximal connected subscraph  $G_0$ ,  $R_0$  and  $\pi(R) \to \pi(G_0)$  and  $\pi(R) \to \pi(G_0)$  and  $G_0$ . But it is the state of  $G_0$  and  $G_0$  is the state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  are state of  $G_0$  are state of  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$  and  $G_0$  are state of  $G_0$ . Regions of  $G_0$  are state of  $G_0$  are state of  $G_0$  are state of  $G_0$  are state of  $G_0$ . Regions of  $G_0$  are state of  $G_0$  are state of  $G_0$  are state of  $G_0$  are state of  $G_0$ . Regions of  $G_0$  are state of  $G_0$  are

there are at least  $\frac{1}{2}\pi R$  points of R which are connected with B by blue edges

Let V denote the set of these points of R and W be the maximal bine-connected subgraph that contains B. If Y is a point such that Y  $\notin$  R and Y  $\notin$  W then YV<sub>i</sub> is yellow for V<sub>i</sub>  $\in$  V. Let Q denote the maximal yellow-connected subgraph that contains Y. If there is no such Y. Q denotes the empty set. R, W, Q contain together all points of G. Namely any points S  $\in$  R is connected with a

<sup>1</sup> The weaker result p(k, l) = k+l can be easily proved. Let us consider any vertex P and a pair of paths of d and  $\overline{0}$  without common vertices except P. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all P and all pairs.) From that the statement follows.

#### Theorem (Gyárfás and Lehel; Faudree and Schelp, 1973)

 $R_{K_{n,n}}(P_n,P_m)\approx n+m$ 

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#### Theorem (Gyárfás and Lehel, 1973)

Let G be a 2-edge-coloured balanced complete bipartite graph. Then one of the following holds.



- G looks like this: X
- Then there are two disjoint monochromatic paths covering all, except possibly one vertex in G.

#### Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

## Conjecture (Gyárfás, 1989)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic paths.

This theorem and conjecture gave rise to a number of results.

# Conjecture (Gyárfás, 1989)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic paths.

- Every *r*-edge-coloured infinite complete graph can be covered by *r* infinite monochromatic paths. [Rado, 1987]
- Every *r*-edge-coloured  $K_n$  can be covered by  $O(r^2 \log r)$  disjoint monochromatic cycles. [Erdős, Gyárfás and Pyber, 1991]
- Every r-edge-coloured K<sub>n</sub> can be covered by O(r log r) disjoint monochromatic cycles. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2006]
- Every 2-edge-coloured K<sub>n</sub> can be covered 2 disjoint monochromatic cycles. [Łuczak, Rödl and Szemerédi, 1998; Allen, 2008; Bessy and Thomassé, 2010]

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#### Conjecture (Gyárfás, 89)

Every r-edge-coloured complete graph can be covered by r disjoint monochromatic paths.

This conjecture led to...

- Every 3-edge-coloured K<sub>n</sub> has 3 monochromatic cycles covering n-o(n) vertices. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2011]
- Not every 3-edge-coloured *K<sub>n</sub>* can be covered by 3 disjoint monochromatic cycles. [P., 2013]
- Every 3-edge-coloured  $K_n$  can be covered by 3 disjoint monochromatic paths. [P., 2013]
- Suppose that we have a sequence G = {G<sub>0</sub>, G<sub>1</sub>, G<sub>2</sub>,...} of graphs with maximum degree ≤ Δ. Every 2-edge-coloured complete graph can be covered by at most 2<sup>C∆ log Δ</sup> monochromatic copies of graphs from G. [Grinshpun and Sárközy, 2013]

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May 22, 2014 12 / 17

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## Results

#### Theorem (P., 2014+)

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Suppose that the edges of  $K_n$  are 2-coloured. Then  $K_n$  can be covered by k disjoint red paths and a disjoint blue balanced complete (k + 1)-partite graph.

#### Theorem (P., 2014+)

Suppose that the edges of  $K_n$  are 2-coloured such that the red subgraph is connected. Then  $K_n$  can be covered by k disjoint red paths and a disjoint blue balanced complete (k + 2)-partite graph.

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#### Theorem (P., 2014+)

Suppose that the edges of  $K_n$  are 2-coloured. Then  $K_n$  can be covered by k disjoint red paths and a disjoint blue balanced complete (k + 1)-partite graph.

• Generalises original Gerencsér-Gyárfás path partitioning theorem.

#### Theorem (P., 2014+)

- Generalises original Gerencsér-Gyárfás path partitioning theorem.
- Can be used to prove the r = 3 case of Gyárfás Conjecture.

#### Theorem (P., 2014+)

• 
$$R(P_n, K_i^t) = (t-1)(n-1) + t(i-1) + 1$$
  
for  $i \equiv 1 \pmod{n-1}$ .

#### Theorem (P., 2014+)

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- $R(P_n, P_n^k) = (n-1)k + \lfloor \frac{n}{k+1} \rfloor$  (Conjectured by Allen, Brightwell and Skokan).
- Might be useful for finding  $R(P_n, H)$  for other graphs  $H_{...?}$

# Proof

#### Theorem

Every 2-edge-coloured complete graph can be covered by a red path and a disjoint blue balanced complete bipartite graph.

Proof.

# Open problems

#### Conjecture

Every 2-edge-coloured complete tripartite graph can be covered by two disjoint monochromatic paths.

#### Conjecture (Gyárfás and Sarközy)

Every complete r-uniform hypergraph H can be covered by  $\alpha(H)$ disjoint loose cycles.

#### Problem

Every r-edge-coloured complete graph can be covered by 1000r monochromatic paths.

. . . . . . . .

# Open problems

#### Problem

Prove natural statements of the form "Every 2-edge-coloured complete graph can be covered by a red graph G and a disjoint blue graph H with G and H having particular structures".

Known results of this type:

- G and H paths [Gerencsér and Gyárfás].
- *G* and *H* cycles [Łuczak, Rödl, and Szemerédi; Allen; Bessy and Thomassé].
- *G* a matching, *H* a complete graph [folklore].
- G a forest of k paths, H a balanced complete (k + 1)-partite graph. [P.]
- G a cycle, H a graph with  $\Delta(H) \geq \frac{1}{2}(|H|-1)$ . [P.]