The discrete Voronoi game

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March 26th, 2015

Competitive facility location

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V

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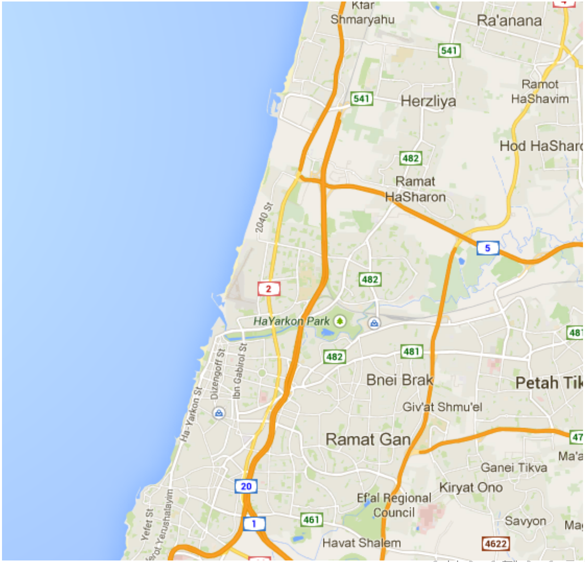
Example:



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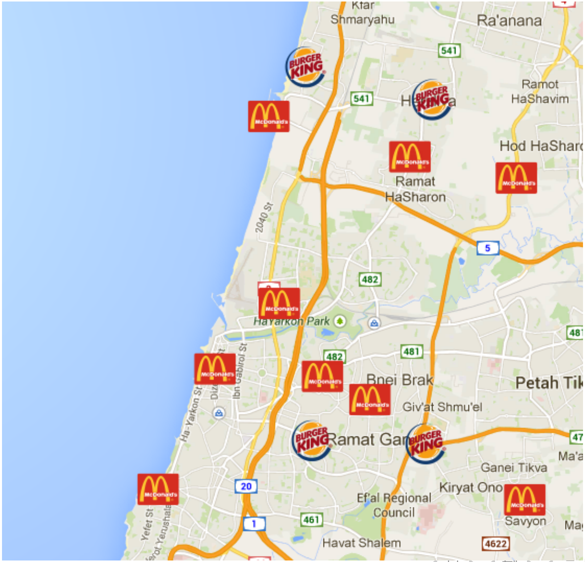

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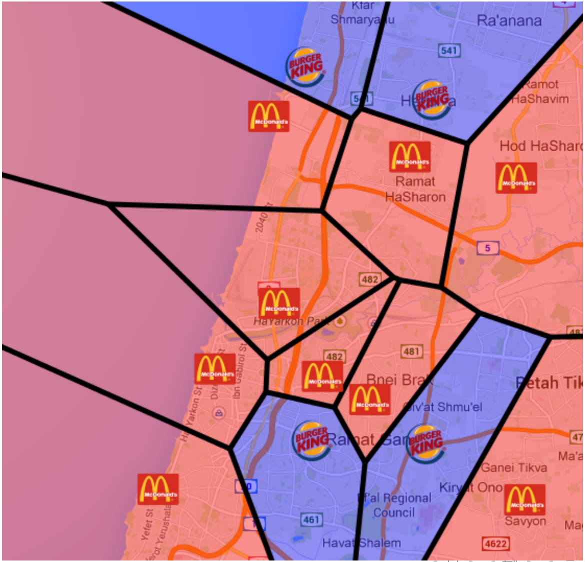

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- The winner is the player with the most points.

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Examples:

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Definition

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- $VR(S_n, t) = 1 - \frac{t}{n}$.
- There are graphs with $VR(G, t) \leq \epsilon$ [Gerbner, Mészáros, Pálvölgyi, P., Rote].

Bounds on the Voronoi ratio

Theorem (Gerbner, Mészáros, Pálvölgyi, P., Rote)

For every graph G we have

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- Both bounds are proved by strategy stealing

Proof sketch

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  - Make the optimal first move in the one round game (which claims $VR(G,1)$ of the vertices.)
  - Follow Player 2's optimal strategy (for $t-1$ moves).
- It is possible to show that playing the last move could not gain more that $VR(G,1)$ of the vertices.

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- For the rest of the game Player 2 just tries to win the subgame on S.

Open problems

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There is an $\alpha > 0$ such that for every planar G we have

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If Player 1 starts by making k simultaneous moves, and then Player 2 makes just one move, then can Player 2 still win 99% of the graph?

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